Causal reasoning with cyclic graphs

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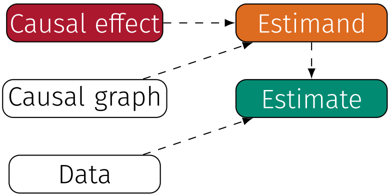
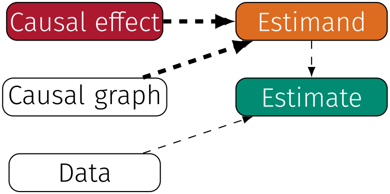
SUMMARY

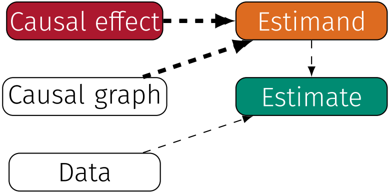
1

The SCM framework

Causal inference and identifiability




Causal effect
Estimand
Causal graph
Estimate
Data



The causal effect is said to be identifiable if it is uniquely computable from $P(\mathbb{V})$.

Total effect a.k.a Average Treatment Effect (ATE)

Total effect

$$= \mathbb{E}(\mathbb{Y} \mid do(\mathbb{X} = \mathbb{x})) - \mathbb{E}(\mathbb{Y} \mid do(\mathbb{X} = \mathbb{x}'))$$

$$\sim P(\mathbb{y} \mid do(\mathbb{x}))$$

Where $do(\cdot)$ is an operator representing an intervention.

Pearl, "Causal diagrams for empirical research". Biometrika, 1995

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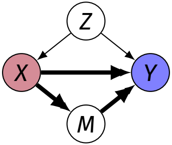

$P(\mathbb{y} \mid do(\mathbb{x}))$ is identifiable if it is uniquely computable from a **positive observationnal** distribution $P(\mathbb{V})$.

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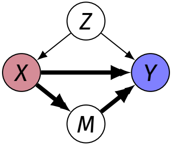

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The **do-calculus** is a **sound and complete** for identification.

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Structural Causal Model

$$
\begin{aligned}
\forall x,\ & \xi_x \leftarrow \mathcal{D}_x \\
A &:= f_A(\xi_a, \xi_{ab}) \\
G &:= f_G(\xi_g) \\
H &:= f_H(G, \xi_h) \\
I &:= f_I(G, \xi_i) \\
B &:= f_B(A, H, \xi_b, \xi_{ab}) \\
C &:= f_C(A, B, I, \xi_c) \\
F &:= f_F(C, G, \xi_f) \\
D &:= f_D(C, F, \xi_d) \\
E &:= f_E(B, D, G, \xi_e)
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Causal graphs

Structural Causal Model

Directed Acyclic Graph (DAG)

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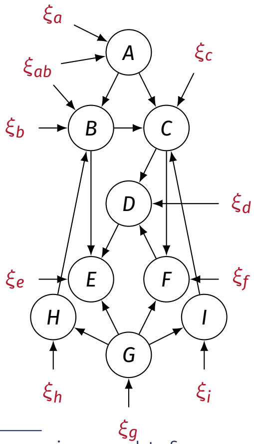

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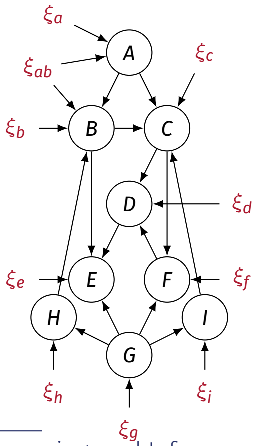
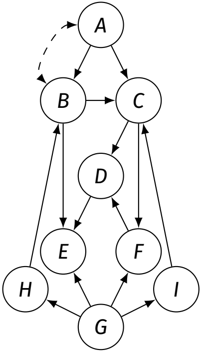


Acyclic Directed Mixed Graph (ADMG)



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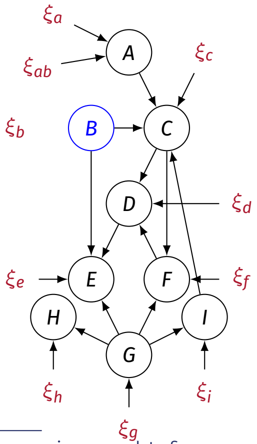

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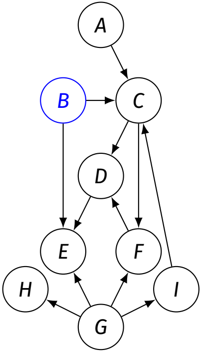

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D-SEPARATION FOR ADMGS

A path is said to be blocked by a set of vertices $\mathbb{Z} \subset \mathbb{V}$ if:

- it contains a chain $\langle A {*\!\!\rightarrow} B \rightarrow C \rangle$ or $\langle A \leftarrow B {\leftarrow\!\!*} C \rangle$ or a fork $\langle A \leftarrow B \rightarrow C \rangle$ and $B \in \mathbb{Z}$; or
- it contains a collider $\langle A {*\!\!\rightarrow} B {\leftarrow\!\!*} C \rangle$ such that no descendant of B is in $\mathbb{Z}$.

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$\mathbb{X}$ and $\mathbb{Y}$ are d-separated by $\mathbb{Z}$ if every path between $\mathbb{X}$ and $\mathbb{Y}$ is blocked by $\mathbb{Z}$ and we write $(\mathbb{X} \perp\!\!\!\perp_d \mathbb{Y} \mid \mathbb{Z})_{\mathcal{G}}$.

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Theorem

$(\mathbb{X} \perp\!\!\!\perp_d \mathbb{Y} \mid \mathbb{Z})_{\mathcal{G}} \Rightarrow \mathbb{X} \perp\!\!\!\perp_{\mathrm{Pr}} \mathbb{Y} \mid \mathbb{Z}$

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DO-CALCULUS FOR ADMGS

The do-calculus consists of three rules:

Rule 1 $P(y|do(x), z, w) = P(y|do(x), w)$ if $(\mathbb{Y} \perp\!\!\!\perp_d \mathbb{Z} \mid \mathbb{X}, \mathbb{W})_{\mathcal{G}_{\overline{\mathbb{X}}}}$

Rule 2 $P(y|do(x, z), w) = P(y|do(x), z, w)$ if $(\mathbb{Y} \perp\!\!\!\perp_d \mathbb{Z} \mid \mathbb{X}, \mathbb{W})_{\mathcal{G}_{\overline{\mathbb{X}}\underline{\mathbb{Z}}}}$

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Theorem

$P(y \mid do(x))$ is identifiable if and only if there exists a finite sequence of transformations, each conforming to either one of the Rules 1-3 or some standard probability manipulations, that reduces $P(y \mid do(x))$ into a do-free formula.

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The do-calculus is sound and complete!

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Back-Door Criterion

Definition

A set of variables $\mathbb{Z}$ satisfies the back-door criterion relative to (X, Y) if:

- $\mathbb{Z} \cap \boldsymbol{De}(X, \mathcal{G}) = \emptyset$, and
- $\mathbb{Z}$ blocks every back-door path (*i.e.*, $\langle X \leftarrow \cdots Y \rangle$).

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Theorem

If $\mathbb{Z}$ satisfies the back-door criterion relative to (X, Y) then:

$$\Pr(y \mid do(x)) = \sum_{\mathbb{z}} \Pr(y \mid x, \mathbb{z}) \Pr(\mathbb{z})$$

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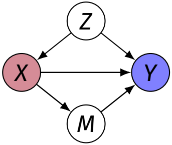

$$
\begin{aligned}
P(y \mid do(x)) \\
&= \sum_z P(y \mid do(x), z) P(z \mid do(x)) \\
&= \sum_z P(y \mid x, z) P(z \mid do(x)) \quad \text{(Rule 2)} \\
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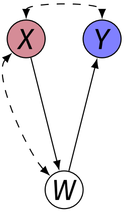
The do-calculus can time be consuming



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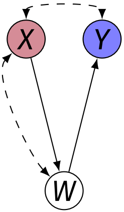
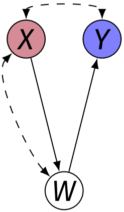
$P(\mathbb{y} \mid do(\mathbb{x}))$



THE DO-CALCULUS CAN TIME BE CONSUMING

$$P(\mathbb{y} \mid do(\mathbb{x})) = \sum_{\mathbb{w}} P(\mathbb{y} \mid do(\mathbb{x}), \mathbb{w}) P(\mathbb{w} \mid do(\mathbb{x}))$$



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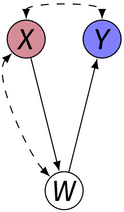

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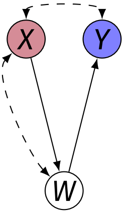

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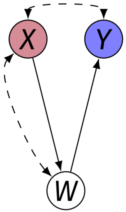
The do-calculus can time be consuming



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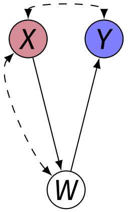

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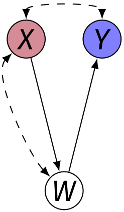

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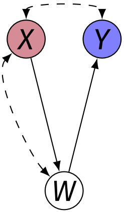

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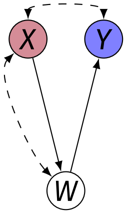
$$= \dots$$

The do-calculus can time be consuming



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&= \sum_{w,x'} P(y \mid w, x') P(x') P(w \mid do(x)) \quad \text{(Rule 3)} \\
&= \ldots
\end{aligned}
$$

In this specific case, the total effect is not identifiable

$\Longrightarrow$ We can never find a do-free formula!

Hedge criterion for ADMGs

A hedge for $P(\mathbb{y} \mid do(\mathbb{x}))$ is a subgraph with specific graphical constraints related to $\mathbb{X}$ and $\mathbb{Y}$.

Shpitser and Pearl, "Complete identification methods for the causal hierarchy". JMLR, 2008

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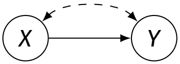
A hedge for $P(\mathbb{y} \mid do(\mathbb{x}))$ is a subgraph with specific graphical constraints related to $\mathbb{X}$ and $\mathbb{Y}$.

- no bidirected dashed edges $\implies$ no hedges
- This graph is a hedge:



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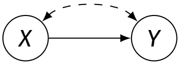

Theorem

$P(\mathbb{y} \mid do(\mathbb{x}))$ is not identifiable in $\mathcal{G}$ <u>if and only if</u> there is a hedge for the ordered pair $(\mathbb{X}, \mathbb{Y})$ in $\mathcal{G}$.

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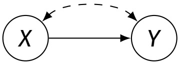
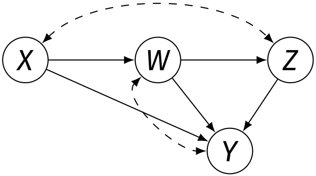
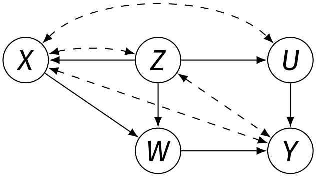
- no bidirected dashed edges $\Longrightarrow$ no hedges
- This graph is a hedge:



Theorem

$P(\mathbb{y} \mid do(\mathbb{x}))$ is not identifiable in $\mathcal{G}$ if and only if there is a hedge for the ordered pair $(\mathbb{X}, \mathbb{Y})$ in $\mathcal{G}$.





Shpitser and Pearl, "Complete identification methods for the causal hierarchy". JMLR, 2008

Hedge criterion for ADMGs: definition

C-component: Set of vertices which are all connected via bidirected arrows.

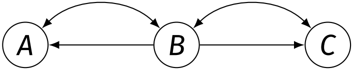


Figure: C-component

Forest: Acyclic graph in which every edge has at most one children.
We call roots the nodes with no children.

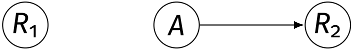


Figure: Forest

Hedge for $(\mathbb{X}, \mathbb{Y})$: Pair of $\mathbb{R}$-rooted C-forests $(\mathcal{F}, \mathcal{F}')$ in the graph such that:

- $\mathbb{R} \subseteq An(\mathbb{Y}, \mathcal{G} \backslash \{V\})$
- $\mathcal{F} \subseteq \mathcal{F}'$
- $\mathcal{F} \cap \mathbb{X} = \emptyset$
- $\mathcal{F}' \cap \mathbb{X} \neq \emptyset$

2

Cluster DMGs over ADMGs

- Definitions
- Summary causal graphs
- Causal tools

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Cluster DMGs over ADMGs

Definitions

Summary causal graphs

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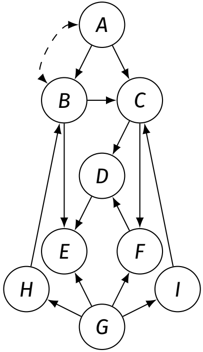
Cluster DMGs over ADMGs

ADMG

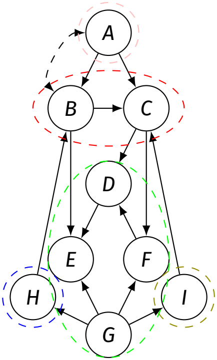


C-DMG over ADMGs

Ferreira and Assaad, Identifying Macro Causal Effects in C-DMGs.

Cluster DMGs over ADMGs

ADMG



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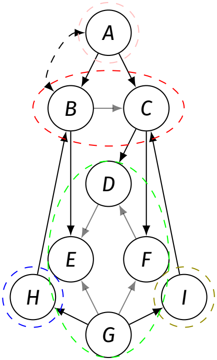

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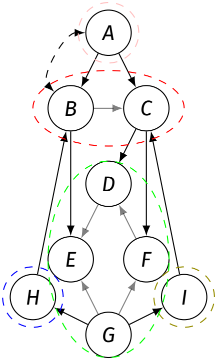
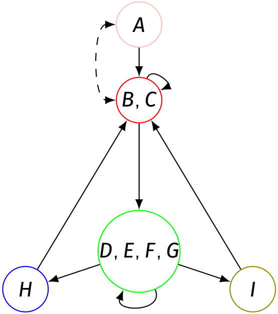
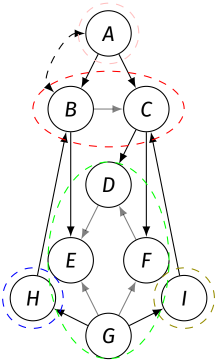


C-DMG over ADMGs



Ferreira and Assaad, Identifying Macro Causal Effects in C-DMGs.

ADMG



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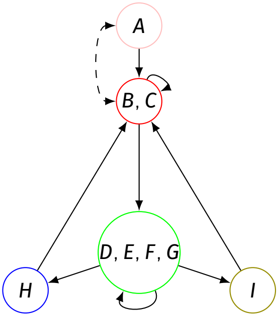
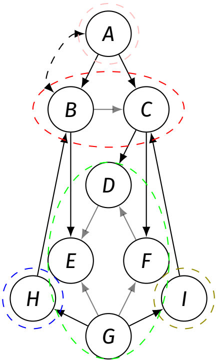


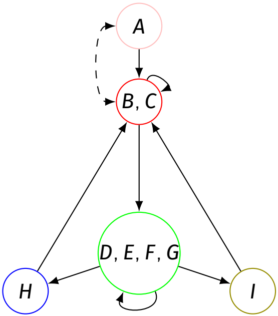
Cycles !

Ferreira and Assaad, Identifying Macro Causal Effects in C-DMGs.

ADMG



C-DMG over ADMGs



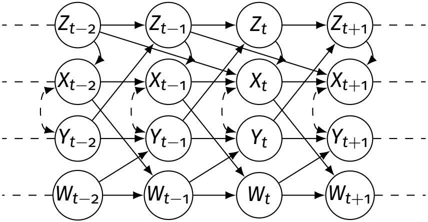
Assumption: For every cycle (e.g., $(B, C) \rightarrow (D, E, F, G) \rightarrow (H) \rightarrow (B, C)$) no 2 adjacent clusters are of size 1.

Ferreira and Assaad, Identifying Macro Causal Effects in C-DMGs.

2

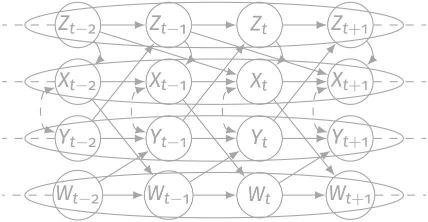
Cluster DMGs over ADMGs

ADMGs with time and summary causal graphs

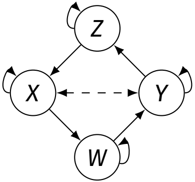


Assaad, Devijver, and Gaussier, "Survey and Evaluation of Causal Discovery Methods for Time Series". JAIR, 2022

ADMGs with time and summary causal graphs



Summary causal graph (SCG)



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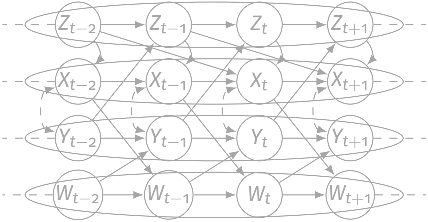
ADMGs WITH TIME AND SUMMARY CAUSAL GRAPHS



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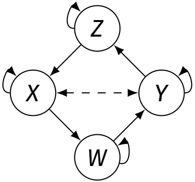

- SCGs can contain cycles

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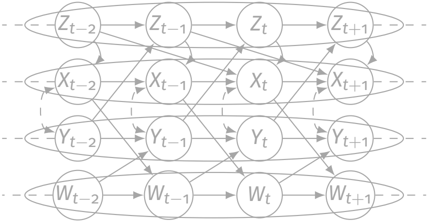
ADMGs with time and summary causal graphs



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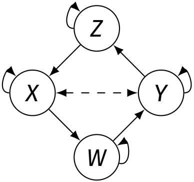
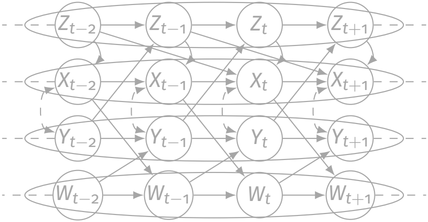


- SCGs can contain cycles
- Each vertex does not map to one single random variable

Assaad, Devijver, and Gaussier, “Survey and Evaluation of Causal Discovery Methods for Time Series”. JAIR, 2022

ADMGs with time and summary causal graphs



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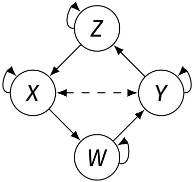

- SCGs can contain cycles
- Each vertex does not map to one single random variable
- There might exists many ADMGs compatible with one SCG

Assaad, Devijver, and Gaussier, “Survey and Evaluation of Causal Discovery Methods for Time Series”. JAIR, 2022

Two types of total effects in C-DMGs

A micro total effect is a total effect from a single variable in a cluster (e.g., $X_{t-\gamma}$) to another single variable (e.g., Y_t). For example: $\Pr(y_t \mid do(x_{t-\gamma}))$.

Ferreira and Assaad, "Identifying macro conditional independencies and macro total effects in summary causal graphs with latent confounding".

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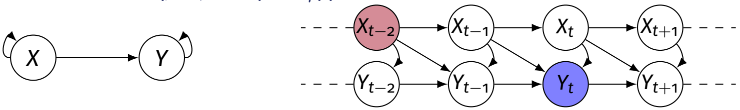

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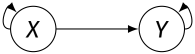

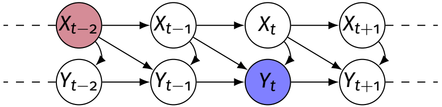

A macro total effect is a total effect from a whole cluster (e.g., $\{X_{t_0}, X_{t_0+1}, \cdots, X_{t-1}, X_t\}$) to a whole other cluster (e.g., $\{Y_{t_0}, Y_{t_0+1}, \cdots, Y_{t-1}, Y_t\}$). For example: $\Pr(\{y_{t_0}, y_{t_0+1}, \cdots, y_{t-1}, y_t\} \mid do(\{x_{t_0}, x_{t_0+1}, \cdots, x_{t-1}, x_t\}))$.

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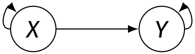

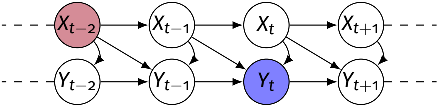

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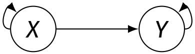

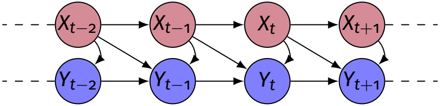

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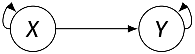

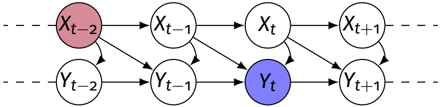

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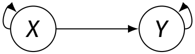

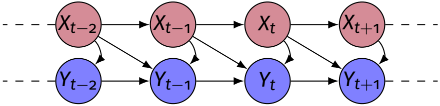

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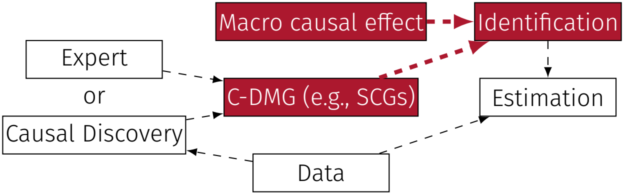
Causal Reasoning



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Cluster DMGs over ADMGs

d-separation for C-DMGs over ADMGs

Is d-separation applicable for C-DMGs over ADMGs?

d-separation for C-DMGs over ADMGs

Is d-separation applicable for C-DMGs over ADMGs?
Yes!

d-separation for C-DMGs over ADMGs

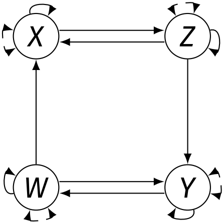
Is d-separation applicable for C-DMGs over ADMGs?
Yes!

Theorem

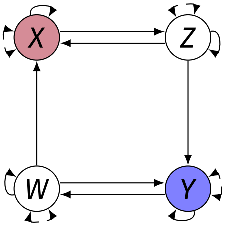
d-separation is valid in C-DMGs over ADMGs.

- *If a d-separation holds in a given C-DMG, then it holds in every compatible ADMG.*
- *If a d-separation does not hold in a given C-DMG, then there exists a compatible ADMG in which it does not hold.*

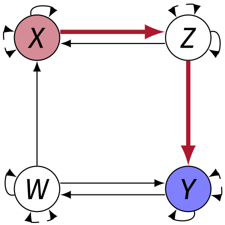
Demo: d-separation for C-DMGs over ADMGs



DEMO: D-SEPARATION FOR C-DMGS OVER ADMGS

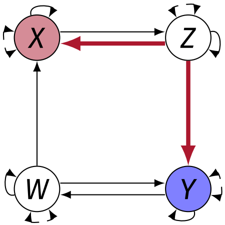


$$X \overset{?}{\perp\!\!\!\perp}_{\mathcal{G}} Y \mid Z, W$$



Is the path $\langle X \rightarrow Z \rightarrow Y \rangle$ blocked by $\{Z, W\}$?

$$X \overset{?}{\perp\!\!\!\perp}_{\mathcal{G}} Y \mid Z, W$$



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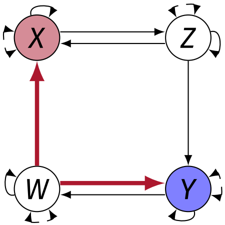

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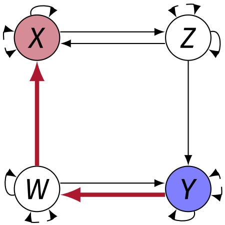

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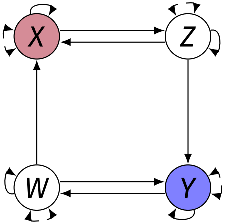

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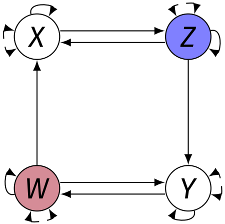
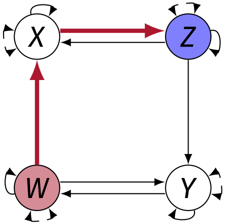


$$X \perp\!\!\!\perp_{\mathcal{G}} Y \mid Z, W$$

$$W \overset{?}{\perp\!\!\!\perp}_{\mathcal{G}} Z \mid \mho \setminus \{X\}$$

Demo: d-separation for C-DMGs over ADMGs



Is the path $\langle W \to X \to Z \rangle$ blocked by $\mho \setminus \{X\}$?

$X \perp\!\!\!\perp_{\mathcal{G}} Y \mid Z, W$

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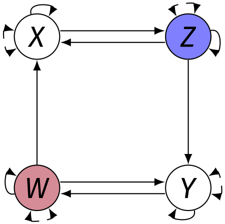

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$W \not\!\perp\!\!\!\perp_{\mathcal{G}} Z \mid \Omega \setminus \{X\}$

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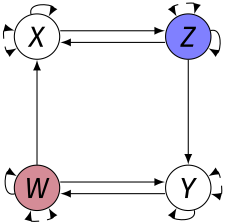
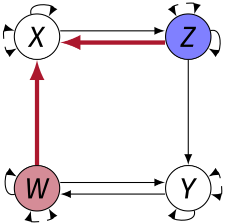


$$X \perp\!\!\!\perp_{\mathcal{G}} Y \mid Z, W$$

$$W \not\perp\!\!\!\perp_{\mathcal{G}} Z \mid \mho \setminus \{X\}$$

$$W \overset{?}{\perp\!\!\!\perp}_{\mathcal{G}} Z \mid \mho \cup \{X\}$$



Is the path $\langle W \rightarrow X \leftarrow Z \rangle$ blocked by $\mho \cup \{X\}$?

$X \perp\!\!\!\perp_{\mathcal{G}} Y \mid Z, W$

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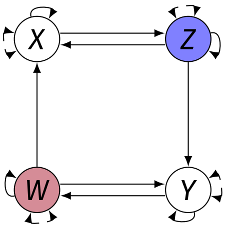
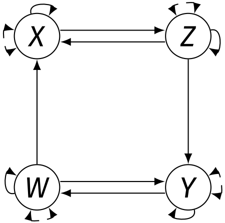


$X \perp\!\!\!\perp_{\mathcal{G}} Y \mid Z, W$

$W \not\!\perp\!\!\!\perp_{\mathcal{G}} Z \mid \mho \setminus \{X\}$

$W \not\!\perp\!\!\!\perp_{\mathcal{G}} Z \mid \mho \cup \{X\}$

Demo: d-separation for C-DMGs over ADMGs



$X \perp\!\!\!\perp_{\mathcal{G}} Y \mid Z, W$

$W \not\perp\!\!\!\perp_{\mathcal{G}} Z \mid \Omega \setminus \{X\}$

$W \not\perp\!\!\!\perp_{\mathcal{G}} Z \mid \Omega \cup \{X\}$

$X \perp\!\!\!\perp_{\mathcal{G}} Y \mid Z, W$

do-calculus for C-DMGs over ADMGs

Is do-calculus applicable for C-DMGs over ADMGs?

do-calculus for C-DMGs over ADMGs

Is do-calculus applicable for C-DMGs over ADMGs?

Yes!

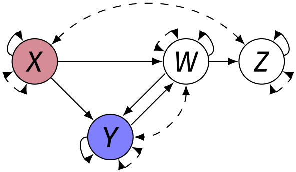
Is do-calculus applicable for C-DMGs over ADMGs?
Yes!

Theorem

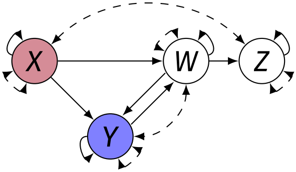
The Rules 1-3 of the do-calculus are valid in C-DMGs over DMGs.

- *If a sequence of rules apply in a given C-DMG, then it applies in every compatible ADMG.*
- *If a sequence of rules of the do-calculus does not apply in a given C-DMG, then there exists a compatible ADMG in which it does not apply.*

Demo: do-calculus for C-DMGs over ADMGs



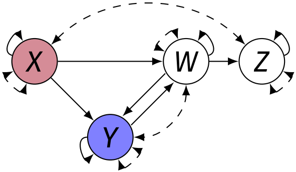
Demo: do-calculus for C-DMGs over ADMGs

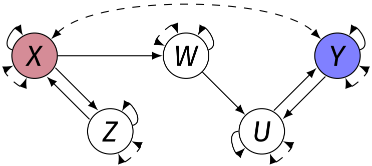


$P(\mathbb{y} \mid do(\mathbb{x}))$

$= P(\mathbb{y} \mid \mathbb{x})$ (Rule 2)

Demo: do-calculus for C-DMGs over ADMGs





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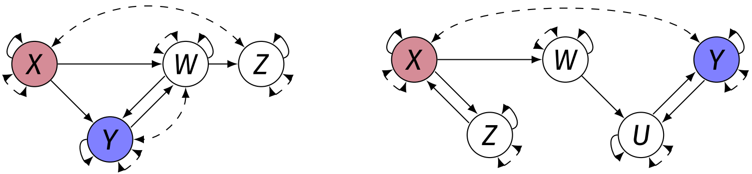

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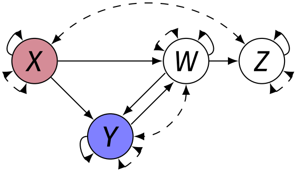
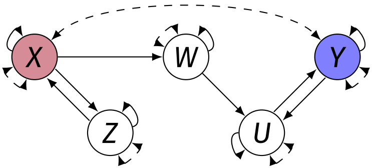
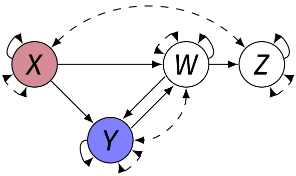


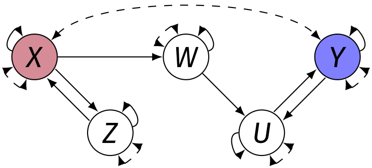


$$P(\mathbb{y} \mid do(\mathbb{x}))$$
$$= \sum_{\mathbb{w}} P(\mathbb{y} \mid do(\mathbb{x}), \mathbb{w}) P(\mathbb{w} \mid do(\mathbb{x}))$$

$$P(\mathbb{y} \mid do(\mathbb{x}))$$
$$= P(\mathbb{y} \mid \mathbb{x}) \quad \text{(Rule 2)}$$

Demo: do-calculus for C-DMGs over ADMGs





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$$= \sum_{\mathbb{w}} P(\mathbb{y} \mid do(\mathbb{x}), do(\mathbb{w})) P(\mathbb{w} \mid \mathbb{x}) \quad \text{(Rule 2) x2}$$

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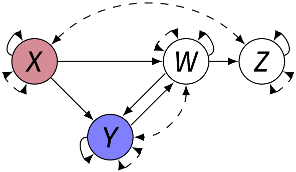
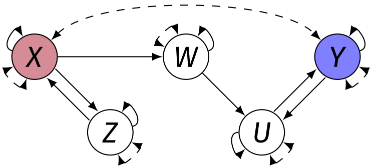
DEMO: DO-CALCULUS FOR C-DMGS OVER ADMGS

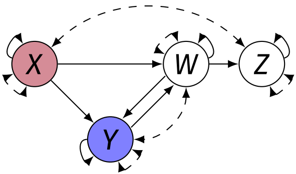




$$P(\mathbb{y} \mid do(\mathbb{x})) = P(\mathbb{y} \mid \mathbb{x}) \quad \text{(Rule 2)}$$

$$\begin{aligned} P(\mathbb{y} \mid do(\mathbb{x})) &= \sum_{\mathbb{w}} P(\mathbb{y} \mid do(\mathbb{x}), \mathbb{w}) P(\mathbb{w} \mid do(\mathbb{x})) \\ &= \sum_{\mathbb{w}} P(\mathbb{y} \mid do(\mathbb{x}), do(\mathbb{w})) P(\mathbb{w} \mid \mathbb{x}) \quad \text{(Rule 2) x2} \\ &= \sum_{\mathbb{w}} P(\mathbb{y} \mid do(\mathbb{w})) P(\mathbb{w} \mid \mathbb{x}) \quad \text{(Rule 3)} \end{aligned}$$

Demo: do-calculus for C-DMGs over ADMGs



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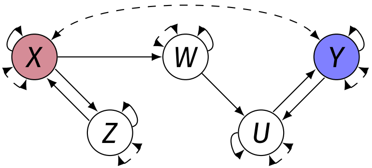

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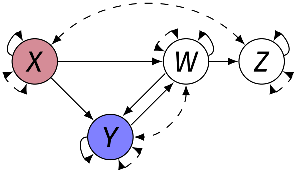

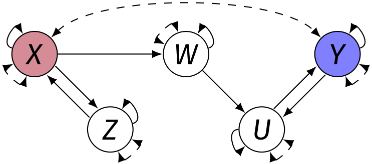

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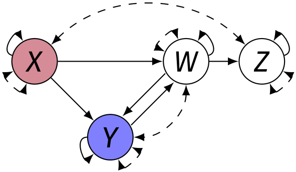

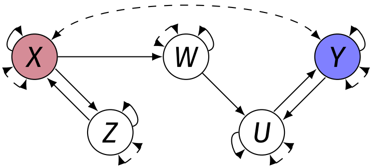

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Completeness of d-separation and do-calculus for C-DMGs over ADMGs

Are the d-separation and do-calculus complete for C-DMGs over ADMGs?

Completeness of d-separation and do-calculus for C-DMGs over ADMGs

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Assumption: For every cycle, no two adjacent clusters are of size 1.

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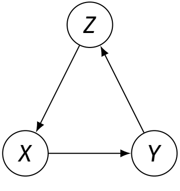

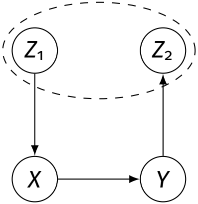

Hedge for C-DMGs over ADMGs

Is the hedge criterion applicable for C-DMGs over ADMGs?

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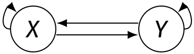

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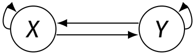

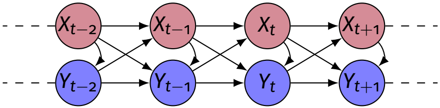

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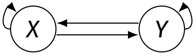

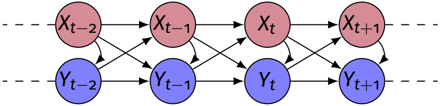

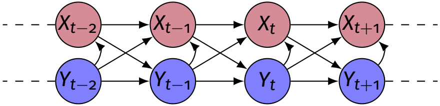

SC-Hedges for C-DMGs over ADMGs

Consider an C-DMG $\mathcal{G}^s$. The SC-projection $\mathcal{H}^s$ of $\mathcal{G}^s$ is the graph that includes all vertices and edges from $\mathcal{G}^s$, plus a bidirected dashed edge between each pair $X, Y \in \mathbb{S}$ such that $X \in Scc(Y, \mathcal{G}) = De(Y, \mathcal{G}^s) \cap An(Y, \mathcal{G}^s)$ and $X \neq Y$.

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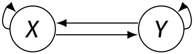

C-DMG

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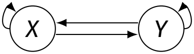
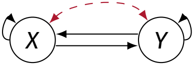


C-DMG



SC-projection

SC-Hedges for C-DMGs over ADMGs

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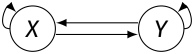
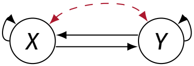


C-DMG



SC-projection

Theorem

Consider that $\mathcal{H}^s$ is the SC-projection of the C-DMG $\mathcal{G}^s$.
$P(\mathbb{y} \mid do(\mathbb{x}))$ *is not identifiable in $\mathcal{G}^s$ if there exists a hedge for the ordered pair $(\mathbb{X}, \mathbb{Y})$ in $\mathcal{H}^s$.*

What about micro-queries?

Reminder:

- A macro causal effect is a causal effect from a whole cluster $\mathbb{X}$ to a another whole cluster $\mathbb{Y}$, e.g., $\Pr(Y_{t_0}, Y_{t_0+1}, \cdots, Y_{t-1}, Y_t \mid do(X_{t_0}, X_{t_0+1}, \cdots, X_{t-1}, X_t))$.
- A micro causal effect is a total effect from a single variable X to another single variable Y, e.g., $\Pr(y_t \mid do(x_{t-\gamma}))$.

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Can we identify micro causal effects in C-DMGs over ADMGs?

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In specific cases and/or with prior knowledge, such as time, then yes!

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- Assaad et al., "Identifiability of total effects from abstractions of time series causal graphs".
- Assaad, "Towards identifiability of micro total effects in summary causal graphs with latent confounding: extension of the front-door criterion".
- Ferreira and Assaad, "Identifiability of Direct Effects from Summary Causal Graphs".
- Ferreira and Assaad, "Average Controlled and Average Natural Micro Direct Effects in Summary Causal Graphs".

3

The Input/Output SCM framework

- Definitions
- Causal tools in DMGs

3

The Input/Output SCM framework

Definitions

Causal tools in DMGs

i/o SCM

$$
\begin{aligned}
\forall x,\ \xi_x &\leftarrow \mathcal{D}_x \\
A &:= f_A(\xi_a, \xi_{ab}) \\
B &:= f_B(A, D, E, \xi_b, \xi_{ab}) \\
C &:= f_C(B, \xi_c) \\
D &:= f_D(C, \xi_d) \\
E &:= f_E(C, \xi_e) \\
(B, C, D) &:= f_{(B,C,D)}(A, E, \xi_b, \xi_{ab}, \xi_c, \xi_d) \\
(B, C, E) &:= f_{(B,C,E)}(A, D, \xi_b, \xi_{ab}, \xi_c, \xi_e) \\
(B, C, D, E) &:= f_{(B,C,D)}(A, \xi_b, \xi_{ab}, \xi_c, \xi_d, \xi_e)
\end{aligned}
$$

Assumption: Compatibility of the generating processes in cycles.

e.g., $(b, c, d) = f_{(B,C,D)}(a, e, \xi_b, \xi_{ab}, \xi_c, \xi_d) \implies b = f_B(a, d, e, \xi_b, \xi_{ab})$

Input/Output SCMs

i/o SCM

~~Acyclic~~ Directed Mixed Graph (DMG)

$$
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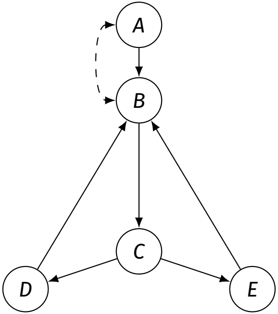

Forré and Mooij, "Causal Calculus in the Presence of Cycles, Latent Confounders and Selection Bias"

Input/Output SCMs

i/o SCM

$$\forall x,\ \xi_x \leftarrow \mathcal{D}_x$$
$$A := f_A(\xi_a, \xi_{ab})$$
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$$C := f_C(B, \xi_c)$$
$$D := d$$
$$E := f_E(C, \xi_e)$$
$$\sout{(B, C, D) := f_{(B,C,D)}(A, E, \xi_b, \xi_{ab}, \xi_c, \xi_d)}$$
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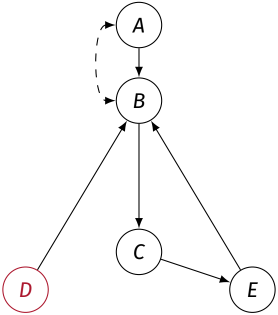
$$\sout{(B, C, D, E) := f_{(B,C,D)}(A, \xi_b, \xi_{ab}, \xi_c, \xi_d, \xi_e)}$$

Directed Mixed Graph (DMG)



Forré and Mooij, "Causal Calculus in the Presence of Cycles, Latent Confounders and Selection Bias"

3

The Input/Output SCM framework

Definitions

Causal tools in DMGs

SIGMA-SEPARATION

Strongly connected component: $\boldsymbol{Scc}(V, \mathcal{G}) := \boldsymbol{An}(V, \mathcal{G}) \cap \boldsymbol{De}(V, \mathcal{G})$

A walk $\tilde{\pi} = \langle V_1, \cdots, V_n \rangle$ is said to be σ-blocked by a set of vertices $\mathbb{Z} \subseteq \mathbb{V}$ if:

1. $\exists 1 < i < n$ such that $\langle V_{i-1} {*\!\!\rightarrow} V_i {\leftarrow\!\!*} V_{i+1} \rangle \subseteq \tilde{\pi}$ and $V_i \notin \mathbb{Z}$, or
2. $\exists 1 < i < n$ such that $\langle V_{i-1} \leftarrow V_i {\leftarrow\!\!*} V_{i+1} \rangle \subseteq \tilde{\pi}$ and $V_i \in \mathbb{Z} \backslash \boldsymbol{Scc}(V_{i-1}, \mathcal{G})$, or
3. $\exists 1 < i < n$ such that $\langle V_{i-1} {*\!\!\rightarrow} V_i \rightarrow V_{i+1} \rangle \subseteq \tilde{\pi}$ and $V_i \in \mathbb{Z} \backslash \boldsymbol{Scc}(V_{i+1}, \mathcal{G})$, or
4. $\exists 1 < i < n$ such that $\langle V_{i-1} \leftarrow V_i \rightarrow V_{i+1} \rangle \subseteq \tilde{\pi}$ and $V_i \in \mathbb{Z} \backslash \left(\boldsymbol{Scc}(V_{i-1}, \mathcal{G}) \cap \boldsymbol{Scc}(V_{i+1}, \mathcal{G}) \right)$.

Forré and Mooij, "Causal Calculus in the Presence of Cycles, Latent Confounders and Selection Bias". PMLR, 2020.

SIGMA-SEPARATION

Strongly connected component: $\boldsymbol{Scc}(V, \mathcal{G}) := \boldsymbol{An}(V, \mathcal{G}) \cap \boldsymbol{De}(V, \mathcal{G})$

A walk $\tilde{\pi} = \langle V_1, \cdots, V_n \rangle$ is said to be σ-blocked by a set of vertices $\mathbb{Z} \subseteq \mathbb{V}$ if:

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4. $\exists 1 < i < n$ such that $\langle V_{i-1} \leftarrow V_i \to V_{i+1} \rangle \subseteq \tilde{\pi}$ and $V_i \in \mathbb{Z} \backslash (\boldsymbol{Scc}(V_{i-1}, \mathcal{G}) \cap \boldsymbol{Scc}(V_{i+1}, \mathcal{G}))$.

$\mathbb{X}$ and $\mathbb{Y}$ are σ-separated by $\mathbb{Z}$ if every walk between $\mathbb{X}$ and $\mathbb{Y}$ is blocked by $\mathbb{Z}$ and we write $(\mathbb{X} \perp\!\!\!\perp_\sigma \mathbb{Y} \mid \mathbb{Z})_\mathcal{G}$.

Forré and Mooij, "Causal Calculus in the Presence of Cycles, Latent Confounders and Selection Bias". PMLR, 2020.

SIGMA-SEPARATION

Strongly connected component: $\boldsymbol{Scc}\left(V, \mathcal{G}\right) := \boldsymbol{An}\left(V, \mathcal{G}\right) \cap \boldsymbol{De}\left(V, \mathcal{G}\right)$

A walk $\tilde{\pi} = \langle V_1, \cdots, V_n \rangle$ is said to be σ-blocked by a set of vertices $\mathbb{Z} \subseteq \mathbb{V}$ if:

1. $\exists 1 < i < n$ such that $\langle V_{i-1} {*\!\!\rightarrow} V_i {\leftarrow\!\!*} V_{i+1} \rangle \subseteq \tilde{\pi}$ and $V_i \notin \mathbb{Z}$, or
2. $\exists 1 < i < n$ such that $\langle V_{i-1} \leftarrow V_i {\leftarrow\!\!*} V_{i+1} \rangle \subseteq \tilde{\pi}$ and $V_i \in \mathbb{Z} \backslash \boldsymbol{Scc}\left(V_{i-1}, \mathcal{G}\right)$, or
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$\mathbb{X}$ and $\mathbb{Y}$ are σ-separated by $\mathbb{Z}$ if every walk between $\mathbb{X}$ and $\mathbb{Y}$ is blocked by $\mathbb{Z}$ and we write $\left(\mathbb{X} \perp\!\!\!\perp_{\sigma} \mathbb{Y} \mid \mathbb{Z}\right)_{\mathcal{G}}$.

Theorem

$\left(\mathbb{X} \perp\!\!\!\perp_{\sigma} \mathbb{Y} \mid \mathbb{Z}\right)_{\mathcal{G}} \Rightarrow \mathbb{X} \perp\!\!\!\perp_{\Pr} \mathbb{Y} \mid \mathbb{Z}$

Forré and Mooij, "Causal Calculus in the Presence of Cycles, Latent Confounders and Selection Bias". PMLR, 2020.

do-calculus for DMGs

The σ-based do-calculus consists of three rules:

Rule 1 $P(\mathbb{y}|do(\mathbb{x}), \mathbb{z}, \mathbb{w}) = P(\mathbb{y}|do(\mathbb{x}), \mathbb{w})$ if $(\mathbb{Y} \perp\!\!\!\perp_\sigma \mathbb{Z} \mid \mathbb{X}, \mathbb{W})_{\mathcal{G}_{\overline{\mathbb{X}}}}$

Rule 2 $P(\mathbb{y}|do(\mathbb{x}, \mathbb{z}), \mathbb{w}) = P(\mathbb{y}|do(\mathbb{x}), \mathbb{z}, \mathbb{w})$ if $(\mathbb{Y} \perp\!\!\!\perp_\sigma \mathbb{Z} \mid \mathbb{X}, \mathbb{W})_{\mathcal{G}_{\overline{\mathbb{X}}\underline{\mathbb{Z}}}}$

Rule 3 $P(\mathbb{y}|do(\mathbb{x}, \mathbb{z}), \mathbb{w}) = P(\mathbb{y}|do(\mathbb{x}), \mathbb{w})$ if $(\mathbb{Y} \perp\!\!\!\perp_\sigma \mathbb{Z} \mid \mathbb{X}, \mathbb{W})_{\mathcal{G}_{\overline{\mathbb{X}}\overline{\mathbb{Z}(\mathbb{W})}}}$

Forré and Mooij, “Causal Calculus in the Presence of Cycles, Latent Confounders and Selection Bias”. PMLR, 2020.

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Rule 3 $P(\mathbb{y}|do(\mathbb{x}, \mathbb{z}), \mathbb{w}) = P(\mathbb{y}|do(\mathbb{x}), \mathbb{w})$ if $(\mathbb{Y} \perp\!\!\!\perp_\sigma \mathbb{Z} \mid \mathbb{X}, \mathbb{W})_{\mathcal{G}_{\overline{\mathbb{X}}\overline{\mathbb{Z}(\mathbb{W})}}}$

Theorem

$P(\mathbb{y} \mid do(\mathbb{x}))$ is identifiable if ~~and only if~~ there exists a finite sequence of transformations, each conforming to either one of the Rules 1-3 or some standard probability manipulations, that reduces $P(\mathbb{y} \mid do(\mathbb{x}))$ into a do-free formula.

Forré and Mooij, "Causal Calculus in the Presence of Cycles, Latent Confounders and Selection Bias". PMLR, 2020.

DO-CALCULUS FOR DMGs

The σ-based do-calculus consists of three rules:

Rule 1 $P(\mathbb{y}|do(\mathbb{x}), \mathbb{z}, \mathbb{w}) = P(\mathbb{y}|do(\mathbb{x}), \mathbb{w})$ if $(\mathbb{Y} \perp\!\!\!\perp_\sigma \mathbb{Z} \mid \mathbb{X}, \mathbb{W})_{\mathcal{G}_{\overline{\mathbb{X}}}}$

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The do-calculus is sound!

Forré and Mooij, "Causal Calculus in the Presence of Cycles, Latent Confounders and Selection Bias". PMLR, 2020.

4

Cluster DMGs over DMGs

- Definitions
- Causal tools

4

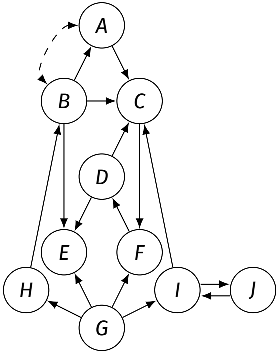
Cluster DMGs over DMGs

Definitions

Causal tools

Cluster DMGs over DMGs

DMG

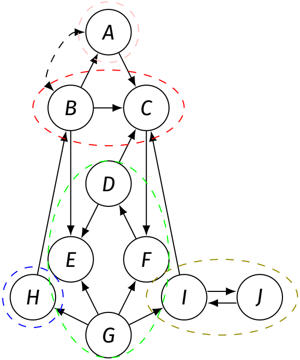


C-DMG over DMGs

Ferreira and Assaad, Identifying Macro Causal Effects in C-DMGs over DMGs.

Cluster DMGs over DMGs

DMG



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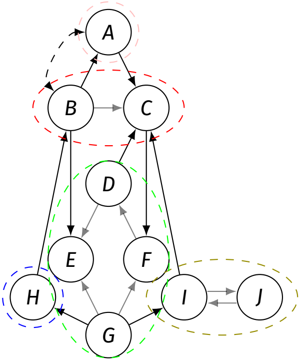
Cluster DMGs over DMGs

DMG



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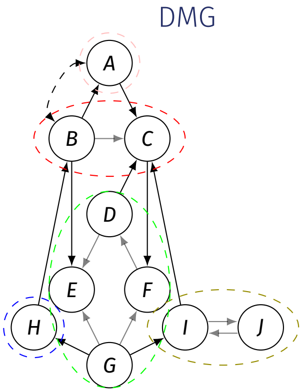
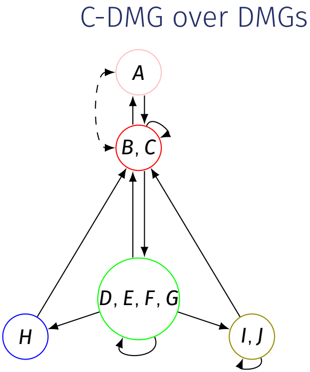
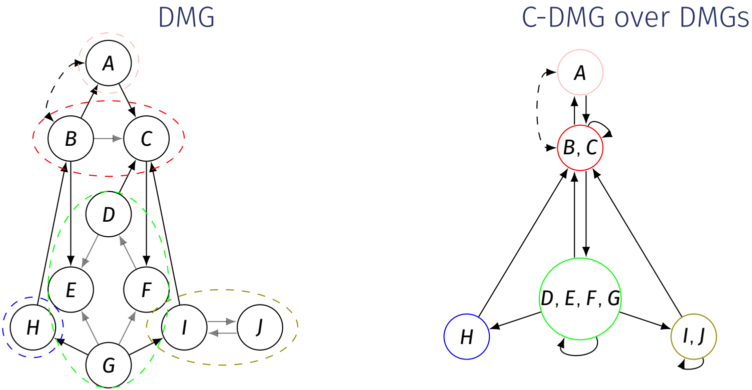
Cluster DMGs over DMGs





Ferreira and Assaad, Identifying Macro Causal Effects in C-DMGs over DMGs.



When clustering, cycles can:

- "disappear" (e.g., $I \to J \to I$) and/or
- "remain" (e.g., $C \to F \to D \to C$) and/or
- "appear" (e.g., $A \to (B, C) \to A$).

Ferreira and Assaad, Identifying Macro Causal Effects in C-DMGs over DMGs.

4

Cluster DMGs over DMGs

Definitions

Causal tools

σ-separation for C-DMGs over DMGs

Is σ-separation applicable for C-DMGs over DMGs?

σ-separation for C-DMGs over DMGs

Is σ-separation applicable for C-DMGs over DMGs?
Yes!

σ-separation for C-DMGs over DMGs

Is σ-separation applicable for C-DMGs over DMGs?
Yes!

Theorem

σ-separation is valid in C-DMGs over DMGs.

- *If a σ-separation holds in a given C-DMG, then it holds in every compatible DMG.*
- *If a σ-separation does not hold in a given C-DMG, then there exists a compatible DMG in which it does not hold.*

σ-based do-calculus for C-DMGs over DMGs

Is σ-based do-calculus applicable for C-DMGs over DMGs?

σ-based do-calculus for C-DMGs over DMGs

Is σ-based do-calculus applicable for C-DMGs over DMGs?

Yes!

σ-based do-calculus for C-DMGs over DMGs

Is σ-based do-calculus applicable for C-DMGs over DMGs?
Yes!

Theorem

The Rules 1-3 of the do-calculus are valid in C-DMGs over DMGs.

- *If a sequence of rules apply in a given C-DMG, then it applies in every compatible DMG.*
- *If a sequence of rules of the do-calculus does not apply in a given C-DMG, then there exists a compatible DMG in which it does not apply.*

5

Conclusion

Key Message

Use causal graphs !

- Even if the causal graph is not fully specified,
- Even if the causal graph is cyclic.

Now, there exist tools which accomodate these cases.

THANK YOU FOR YOUR ATTENTION!

ANY QUESTIONS?

Inserm

iPLesp

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